1 Expected Values

The expected value of a sample $x$ is

$$E[X] = \sum_{Q} P(X = x)x$$

(1)

An expected value is analogous to a mean, except you don’t add data points. You add the elements in your sample space, weighted by their probability. Be careful not to be confused: the expected value is not the most likely outcome. The most likely outcome is the $x$ which maximizes $P(x)$, written as:

$$\text{argmax } x P(x)$$

(2)

Think of an expected value as a measure of the center point of the sample space.

1.1 Die Roll Example

Consider a fair die example, where $d$ is an element in our sample space. Our sample space is one through six and the probability of a sample occurring is $1/6$.

$$E[D] = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 \ldots$$

$$E[D] = \frac{1}{6} \times 21 = 3.5$$

Notice that the expected value is NOT an element of the sample space.

1.2 Unfair Die Example

Now consider our unfair die, where

$$P(D = d) = \frac{d}{21}$$

so for example, the probability of rolling a 4 is $4/21$.

$$E[D] = \sum_{1, \ldots, 6} \frac{d}{21} \times d = \frac{91}{21}$$
2 Expected Value of a Random Variable

To calculate the expected value of a random variable $S$, use:

$$E[S] = \sum_Q P(X = x)S(x)$$ (3)

Remember that the random variable $S$ is defined by some $S(x)$ function, whose input is an element in the sample space and has an output of some real number.

2.1 Expected Value for Sum of 2 Dice

Consider the product sample space of two dice: $Q_2 = Q \otimes Q$, where $Q$ is 1, . . . , 6. Example of elements of this sample space are rolling a 2 for the first die and a 4 for the second die: (2, 4). There are 36 elements in the sample space and we’ll take the probability of each to be equivalent. This gives:

$$P(X = x) = P(X = (x_1, x_2)) = \frac{1}{36}$$

Define $S$ as our random variable representing the sum of the two dice.

$$S(x) = S((x_1, x_2)) = x_1 + x_2$$

Now we can plug these into our expected value equation:

$$E[S] = \sum_Q P(X = x)S(x) = \sum_{(x_1, x_2) \in Q_2} \frac{1}{36}S((x_1, x_2))$$

$$= \frac{1}{36} \times (1 + 1) + \frac{1}{36} \times (1 + 2) + \frac{1}{36} \times (1 + 3) + \ldots + \frac{1}{36} \times (6 + 6)$$

To calculate the answer, we need to sum all 36 possible dice rolls together. That turns out to be 252. So we have:

$$E[S] = \frac{252}{36} = 7$$

2.2 Expected Value for Sum of 2 Dice - Alternate

Another way to solve this problem is to DEFINE our sample space to be the possible sums. Then $Q = \{2, 3, \ldots, 12\}$. Our corresponding probability is going to be the number of permutations to roll the sum divided by some normalizing constant. For example, the probability of rolling a 6 is $5 / Z$, where $Z$ is some constant to make sure our probabilities sum to 1. The 5 comes from the number of ways to roll a 6: (1, 5), (2, 4), (3, 3), (4, 2), (5, 1). I’m calling these permutations because (1, 5) and (5, 1) are both counted. Now we to find $Z$:

$$Z = \underbrace{\frac{1}{2}} + \underbrace{\frac{2}{3}} + \underbrace{\frac{3}{4}} + \underbrace{\frac{4}{5}} + \underbrace{\frac{5}{6}} + \underbrace{\frac{6}{7}} + \underbrace{\frac{7}{8}} + \underbrace{\frac{8}{9}} + \underbrace{\frac{9}{10}} + \underbrace{\frac{10}{11}} + \underbrace{\frac{11}{12}} = 36$$

where the underbraces indicate which dice roll sum each term corresponds to. Doing this work also reveals an equation for the probability:

$$P(S = s) = \frac{6 - |s - 7|}{36}$$
Finally, we are ready to utilize the expected value equation:

\[
E[S] = \sum_{Q} P(S = s)s = \sum_{2,\ldots,12} \frac{6 - |s - 7|}{36} \times s
\]

\[= \frac{1 \times 2}{36} + \frac{2 \times 3}{36} + \frac{3 \times 4}{36} + \ldots + \frac{1 \times 12}{36}
\]

\[= \frac{252}{36} = 7\]

Thus we’ve arrived at the same answer.

2.3 Continuous Expected Value

\[E[X] = \int_{X} x p(x) \, dx \quad (4)\]

2.4 Continuous Example

\[p'(x) \propto x, \quad Q = [0, 5]\]

First, we must normalize it:

\[\int_{0}^{5} x \, dx = \frac{25}{2}\]

\[p(x) = \frac{2x}{25}\]

\[\int_{0}^{5} x \times \frac{2x}{25} = \frac{2 \times 125}{25 \times 3} = \frac{10}{3}\]

2.5 Conditional Expectation Value

\[E(X \mid Y = y) = \sum_{x'} P(X = x \mid Y = y) x \quad (5)\]

2.6 Die Roll Example

Our random variable \(X\) is the observation, and \(Y = 0\) if the observation is odd and \(Y = 1\) if the observation is even.

\[E(X \mid Y = 0) = \frac{1}{3} \times 1 + \frac{1}{3} \times 3 + \frac{1}{3} \times 5 = 3\]

Expected value gives the center of a random variable or probability distribution.

3 Variance

Variance gives the average deviation from that center. To make sure variation above and below the center contributes, it is the expected squared distance from the expected value:

\[\text{Var}(X) = E[(E[X] - X)^2] = E[X^2] - E[X]^2 \quad (6)\]
3.1 Die Roll Example

We already know $E[X] = 21/6 = 3.5$. To find $E[D^2]$, we can use our random variable expectation equation. Take $F$ to be our random variable defined by $F(x) = x^2$:

$$E[D^2] = E[F] = \sum P(x)x^2 =$$

$$= \frac{1}{6} \times 1^2 + \frac{1}{6} \times 2^2 + \frac{1}{6} \times 3^2 \ldots$$

$$= \frac{91}{6}$$

$$\text{Var}(D) = \frac{91}{6} - \frac{21^2}{6^2} = \frac{35}{12} = 2.92$$